3

Answers and Hints

3.7 The Tobacco Market

(a) Let's not use numbers until we need to. We can write $p = a - bq$, where $q = q_1 + q_2 + q_3$, and q_i is the amount sent to market by farmer i. Farmer 1 maximizes $pq_1 = (a - bq)q_1$. If there is an interior solution, the first order condition on q_1 must satisfy

$$a - b(q_2 + q_3) - 2bq_1 = 0.$$

If all farmers ship the same amount of tobacco, then $q_2 = q_3 = q_1$, so this equation becomes $4bq_1 = a$, which gives $q_1 = q_2 = q_3 = a/4b$, and $q = 3a/4b$, so $p = a/4$. The profit of each farmer is $pq = a^2/16b$. In our case $a = 10$ and $b = 1/100000$, so the price is \$2.50 per pound, and each farmer ships 250,000 pounds and discards the rest. The price support doesn't matter, since $p >$ \$0.25. Each farmer has profit \$625,000.

If the second and third farmers send their whole crop to market, then $q_2 + q_3 = 1,200,000$ in this case even if Farmer 1 shipped nothing, the market price would be $10 - 1,2000,000/100,000 = -2 < 0$, so the price support would kick in. Farmer 1 should then also ship all his tobacco at \$0.25 per pound, and each farmer has profit \$150,000.

(b) There are lots. For instance, any combination of quantities for the three farmers that adds up to 750,000 pounds gives the price \$2.50. Even if one farmer shipped the whole crop and the other two each shipped 75,000 pounds, they would make \$187,500 each, which is better than shipping the whole crop.

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3.9 The Klingons and the Snarks

Suppose the Klingons choose a common rate r of consumption. Then each eats 500 snarks, and each has payoff

$$u = 2000 + 50r - r^2.$$

Setting the derivative u' to zero, we get $r = 25$, so each has utility $u = 2000 + 50(25) - 25^2 = 2625$.

Now suppose they choose their rates separately. Then

$$u_1 = \frac{4000r_1}{r_1 + r_2} + 50r_1 - r_1^2.$$

Setting the derivative of this to zero, we get the first order condition

$$\frac{\partial u_1}{\partial r_1} = \frac{4000r_2}{(r_1 + r_2)^2} + 50 - 2r_1 = 0,$$

and a symmetrical condition holds for the second Klingon:

$$\frac{\partial u_2}{\partial r_2} = \frac{4000r_1}{(r_1 + r_2)^2} + 50 - 2r_2 = 0.$$

These two imply

$$\frac{r_2}{r_1} = \frac{r_1 - 25}{r_2 - 25},$$

which has solutions $r_1 = r_2$ and $r_1 + r_2 = 25$. The latter, however, cannot satisfy the first order conditions. Setting $r_1 = r_2$ we get

$$\frac{4000}{r_1} + 50 - 2r_1 = 0,$$

or $1000/r_1 + 50 - 2r_1 = 0$. This is a quadratic that is easy to solve. Multiply by r_1, getting $2r_1^2 - 50r_1 - 1000 = 0$, with solution $r = (50 + \sqrt{(2500 + 8000)})/4 = 38.12$. So the Klingons eat about 50% faster than they would if they cooperated! Their utility is now $u = 2000 + 50r_1 - r_1^2 = 452.87$, much lower than if they cooperated.

3.10 Chess—The Trivial Pastime

(a) The game in §3.9 is not finite, and is not a game of perfect information. The game of Throwing Fingers (§4.10) has no pure strategy Nash equilibrium.

(b) We will have to prove something more general. Let's call a game *Chessian* if it is a finite game of perfect information in which players take turns, and the outcome is either (win,lose), (lose,win), or (draw,draw), where win is preferred to draw, and draw is preferred to lose. Let us call a game *certain* if it has a solution in pure strategies. If a Chessian game is certain, then clearly either one player has a winning strategy, or both players can force a draw. Suppose there were a Chessian game that is not certain. Then there must be a *smallest* Chessian game that is not certain (i.e., one with fewest nodes). Suppose this has k nodes. Clearly $k > 1$, since it is obvious that a Chessian game with one node is certain. Take any node all of whose child nodes are terminal nodes (why must this exist?). Call this node A. Suppose Red (player 1) chooses at A (the argument is similar if Black chooses). If one of the terminal nodes from A is (win,lose), label A (lose,win); if all of the terminal nodes from A are (lose,win), label A (win,lose); otherwise label A (draw,draw). Now erase the branches from A, along with their terminal nodes. Now we have a new, smaller, Chessian game, which is certain, by our induction assumption. It is easy to see that if Red has a winning strategy in the smaller game, it can be extended to a winning strategy in the larger game. Similarly, if Black has a winning strategy in the smaller game, it can be extended to a winning strategy in the larger game. Finally, if both players can force a draw in the smaller game, their respective strategies must force a draw in the larger game.

3.11 The Samaritan's Dilemma

The father's first order condition is

$$U_t(t, s) = -u'(y - t) + \alpha\delta v_2'(s(1 + r) + t) = 0,$$

and the father's second order condition is

$$U_{tt} = u''(y - t) + \alpha\delta v_2''(s(1 + r) + t) < 0.$$

If we treat t as a function of s (backward induction!), then the equation $U_t(t(s), s) = 0$ is an identity, so we can differentiate totally with respect to s, getting

$$U_{tt}\frac{dt}{ds} + U_{ts} = 0.$$

But $U_{ts} = \alpha\delta(1+r)v_2'' < 0$, so $t'(s) < 0$; i.e., the more the daughter saves, the less she gets from her father in the second period.

The daughter's first order condition is

$$v_s(s,t) = -v_1' + \delta v_2'(1 + r + t'(s)) = 0.$$

Suppose the daughter's optimal s is s^*, and so the father's transfer is $t^* = t(s^*)$. If the father precommits to t^*, then $t'(s) = 0$ would hold in the daughter's first order condition. Therefore in this case $v_s(s^*, t^*) > 0$, so the daughter is better off by increasing s to some $s^{**} > s^*$, and hence the father is better off as well, since he is a partial altruist.

For the example, if $u(\cdot) = v_1(\cdot) = v_2(\cdot) = \ln(\cdot)$, then it is straightforward to check that

$$t^* = \frac{y(1 + \alpha\delta(1+\delta)) - \delta(1+r)z}{(1+\delta)(1+\alpha\delta)}$$

and

$$s^* = \frac{\delta(1+r)z - y}{(1+r)(1+\delta)}.$$

If the father can precommit, solving the two first order conditions for maximizing $U(t,s)$ gives

$$t^f = \frac{\alpha(1+\delta)y - (1+r)z}{1 + \alpha + \alpha\delta},$$

$$s^f = \frac{(1+r)(1+\alpha\delta)z - \alpha y}{(1+r)(1+\alpha+\alpha\delta)}.$$

We then find

$$t^* - t^f = \frac{y + (1+r)z}{(1+\delta)(1+\alpha\delta)(1+\alpha+\alpha\delta)} > 0$$

$$s^f - s^* = \frac{y + (1+r)z}{(1+r)(1+\delta)(1+\alpha+\alpha\delta)} > 0.$$

3.12 The Rotten Kid Theorem

(a) Mom's objective function is

$$V(t,a) = u(y(a) - t) + \alpha v(z(a) + t),$$

so her first order condition is

$$V_t(t,a) = -u'(y(a)-t) + \alpha v'(z(a)+t) = 0$$

If we treat t as a function of a in the above equation (there's where backward induction comes in!), it becomes an identity, so we can differentiate with respect to a, getting

$$-u''(y'-t') + \alpha v''(z'+t') = 0. \tag{3.1}$$

Therefore $z'+t'=0$ implies $y'-t' == y'+z'=0$. Thus the first order conditions for the maximization of $z+t$ and $z+y$ have the same solutions.

(b) Notice that since a satisfies $z'(a)+y'(a)=0$, a does not change when α changes. Differentiating the first order condition $V_t(t(\alpha)) = 0$ totally with respect to α, we get

$$V_{tt}\frac{dt}{d\alpha} + V_{t\alpha} = 0.$$

Now $V_{tt} < 0$ by the second order condition for a maximum, and

$$V_{t\alpha} = v' > 0,$$

which proves that $dt/d\alpha > 0$. Since a does not depend on $\hat{y}$, differentiating $V_t(t(y)) = 0$ totally with respect to $\hat{y}$, we get

$$V_{tt}\frac{dt}{d\hat{y}} + V_{t\hat{y}} = 0.$$

But $V_{t\hat{y}} = -u'' > 0$ so $dt/d\hat{y} > 0$.

(c) Suppose t remains positive as $\alpha \to 0$. Then v' remains bounded, so $\alpha v' \to 0$. From the first order condition, this means $u' \to 0$, so $y - t \to \infty$. But y is constant, since a maximizes $y+z$, which does not depend on α. thus $t \to -\infty$.

3.13 The Illogic of Conflict Escalation

We do only the last part of the problem. Each player in effect faces a decision problem against Nature where staying in round k wins \$$v$ with probability p, loses k with probability $p(1-p)$, and leads to round $k+2$ with probability $(1-p)^2$. Dropping in round k for $k > 2$ costs $k-2$.

The pure strategies for the players are s_k, which says "bid until round k, then drop if you have not already won or lost." We calculate the expected payoff to s_k. Consider player 1, who bids first. The probability the game terminates spontaneously at round n is $(1-p)^{n-1}p$. Player 1 wins $\$v$ if the game terminates spontaneously on an odd round $n < k$, loses $n-1$ if the game terminates spontaneously on an even round $n < k$, and loses $k-2$ if the game continues to round k. The expected return from winning 0 for $k = 1$, and for $k \geq 3$ is given by

$$v_{+k} = v(p + p(1-p)^2 + \ldots + p(1-p)^{k-3}) = v\frac{(1-(1-p)^{k-1})}{2-p}.$$

Note that this also holds for $k = 1$. The expected cost of losing by spontaneous termination is

$$v_{-k} = p(1-p) + 3p(1-p)^3 + \ldots + (k-2)p(1-p)^{k-2}.$$

To calculate this in closed form, let

$$f(k,q) = kq^{k+1} + (k+2)q^{k+3} + (k+4)q^{k+5} + \ldots,$$

where $0 < q < 1$, so $v_{-k} = p(1-p)[f(1,1-p) - f(k,1-p)]$. Using $q^2 = r$, we can write

$$\begin{aligned}
f(k,q) &= kq^{k+1}(1 + q^2 + q^4 + \ldots) + 2q^{k+3}(1 + 2q^2 + 3q^4 + \ldots) \\
&= \frac{kq^{k+1}}{1-q^2} + 2q^{k+3}(1 + 2r + 3r^2 + \ldots) \\
&= \frac{kq^{k+1}}{1-q^2} + 2q^{k+3}\frac{d}{dr}(r + r^2 + r^3 + \ldots) \\
&= \frac{kq^{k+1}}{1-q^2} + 2q^{k+3}\frac{d}{dr}(\frac{1}{1-r} - 1) \\
&= \frac{kq^{k+1}}{1-q^2} + \frac{2q^{k+3}}{(1-r)^2} = \frac{kq^{k+1}}{1-q^2} + \frac{2q^{k+3}}{(1-q^2)^2} \\
&= q^{k+1}\frac{k(1-q^2)+2q^2}{(1-q^2)^2} = q^{k+1}\frac{k-(k-2)q^2}{(1-q^2)^2}.
\end{aligned}$$

Thus

$$v_{-k} = (1-p)^2\frac{(2-2p+p^2) - (1-p)^{k-1}(k-(k-2)(1-p)^2)}{p(2-p)^2}.$$

Finally, the expected loss from dropping out at round k is $k-2$ times the probability of reaching round k, or $(k-2)(1-p)^{k-1}$. Thus the total value of strategy s_k for $k > 1$ is

$$v\frac{(1-(1-p)^{k-1})}{(2-p)} - (k-2)(1-p)^{k-1}$$
$$-(1-p)^2\frac{(2-2p+p^2)-(1-p)^{k-1}(k-(k-2)(1-p)^2)}{p(2-p)^2}.$$

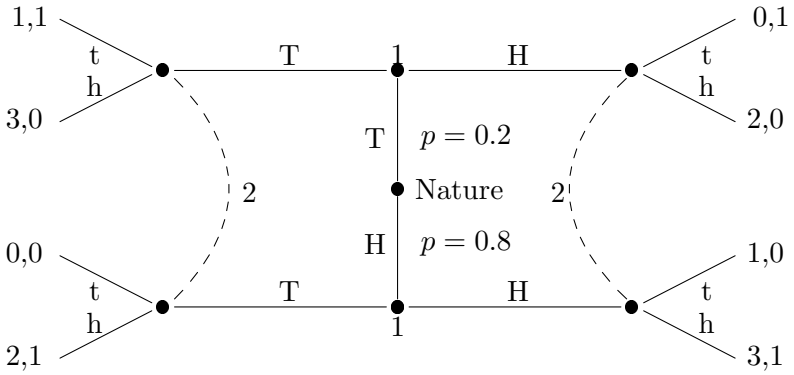
3.14.2 Where's Jack?

Let A be the event "Manny wins on the first round," B the event "Moe wins on the second round," and C the event "A wins on the third round." We have $p(A) = 0.3$, $p(B) = (1-p(A))(0.5) = 0.35$, and $p(C) = (1-p(B)-p(A))(0.4) = 0.14$. Thus the probability Manny wins is $p(A) + p(C) = 0.44$, so Manny's expected payoff is \$44.

3.19 The Truth Game

(a) obvious

(b) Here is the game tree, written 'sideways':



Player 1 has strategy set {HH,HT,TH,TT} where HH means announce H if you see H, announce H if you see T, HT means announce H if you see H, announce T if you see T, etc. Thus HH = always say H, HT = tell the truth, TH = lie, TT = always say T. Player 2 has strategy set {hh,ht,th,tt}, where hh means say h if you are told

h, and say h if you are told t, ht means say h if you are told h and say t if you are told t, etc. Thus hh means always say h, ht means trust Player 1, th means distrust Player 1, and tt means always say t.

(c) Here are the payoffs to the two cases, according to Nature's choice:

	hh	ht	th	tt
HH	2,0	2,0	0,1	0,1
HT	3,0	1,1	3,0	1,1
TH	2,0	2,0	0,1	0,1
TT	3,0	1,1	3,0	1,1

Payoff when coin is T

	hh	ht	th	tt
HH	3,1	3,1	1,0	1,0
HT	3,1	3,1	1,0	1,0
TH	2,1	0,0	2,1	0,0
TT	2,1	0,0	2,1	0,0

Payoff when coin is H

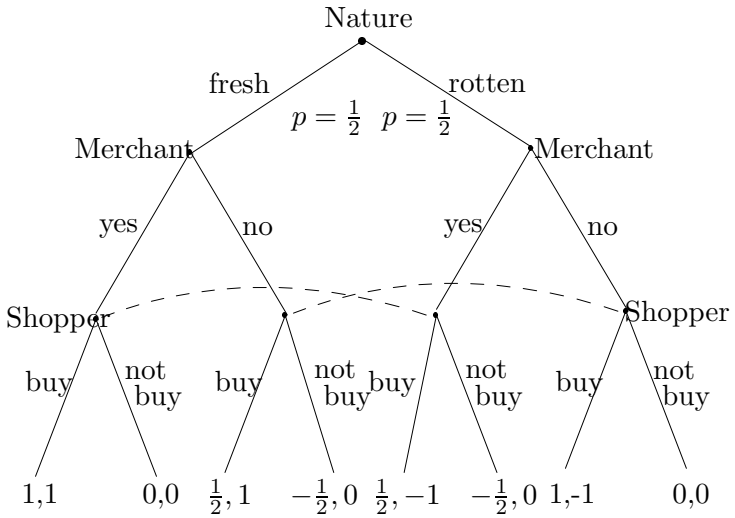
The actual payoff matrix is 0.2 x first matrix + 0.8 x second:

	hh	ht	th	tt
HH	2.8,0.8	2.8,0.8	0.8,0.2	0.8,0.2
HT	3.8,0.8	2.6,1.0	1.2,0.0	1.0,0.2
TH	2.0,0.8	0.4,0.0	1.6,1.0	0.0,0.2
TT	2.2,0.8	0.2,0.8	2.2,0.8	0.2,0.2

(d) tt is dominated by hh, so we can drop tt. There are no other dominated pure strategies.

(e) (TT,th) and (HH,ht) are both Nash. In the first, 1 always says T and 2 assumes 1 lies; in the second, 1 always says H and 2 always believes 1. The first equilibrium is Pareto-inferior to the second.

3.20 The Shopper and the Fish Merchant



Here is the normal form for each of the two cases Good Fish/Bad Fish, and their expected value:

	bb	bn	nb	nn
yy	1,1	1,1	0,0	0,0
yn	1,1	1,1	0,0	0,0
ny	$\frac{1}{2}, 1$	$-\frac{1}{2}, 0$	$\frac{1}{2}, 1$	$-\frac{1}{2}, 0$
nn	$\frac{1}{2}, 1$	$-\frac{1}{2}, 0$	$\frac{1}{2}, 1$	$-\frac{1}{2}, 0$

Good Fish

	bb	bn	nb	nn
yy	$\frac{1}{2}, -1$	$\frac{1}{2}, -1$	$-\frac{1}{2}, 0$	$-\frac{1}{2}, 0$
yn	1,-1	0,0	1,-1	0,0
ny	$\frac{1}{2}, -1$	$\frac{1}{2}, -1$	$-\frac{1}{2}, 0$	$-\frac{1}{2}, 0$
nn	1,-1	0,0	1,-1	0,0

Bad Fish

	bb	bn	nb	nn
yy	$\frac{3}{4}, 0$	$\frac{3}{4}, 0$	$-\frac{1}{4}, 0$	$-\frac{1}{4}, 0$
yn	1,0	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, -\frac{1}{2}$	0,0
ny	$\frac{1}{2}, 0$	$0, -\frac{1}{2}$	$0, \frac{1}{2}$	$-\frac{1}{2}, 0$
nn	$\frac{3}{4}, 0$	$-\frac{1}{4}, 0$	$\frac{3}{4}, 0$	$-\frac{1}{4}, 0$

$\frac{1}{2}$ Good Fish + $\frac{1}{2}$ Bad Fish

Applying the successive elimination of dominated strategies, we have *yn* dominates *ny*, after which *bn* dominates *bb*, *nb*, and *nn*. But then *yy*

dominates yn and nn. Thus a Nash equilibrium is yy/bn: the merchant says the fish is good no matter what, and the buyer believes him. Since some of the eliminated strategies were only weakly dominated, there could be other Nash equilibria, and we should check for this. Certainly ny is strictly dominated. Moreover, if the seller uses yn with positive probability, then the buyer can't use anything but bn, so the seller should *not* use yn. Thus the seller must use yn with zero probability in any Nash equilibrium (truth-telling is never profitable). But then the seller always uses only yy and nn, and all strategies for the buyer have equal payoff (zero). Thus another set of Nash equilibria are for the seller to use any combination of yy and nn, and the buyer to use the pure strategy bb. Yet another is for the seller to use pure strategy nn and the buyer to use pure strategy ny. Note that all these Nash equilibria have the same payoff for the players. Note also that the equilibria using weakly dominated strategies only work if the buyer is a 'nice guy' in the sense of choosing among equally good responses that which maximizes the payoff to the seller. The equilibrium yy/bn does not have this drawback.

3.22 The Women of Sevitan

There are six philandering husbands, and all are branded on the sixth day. The reason this happens is that the women all knew there was at least one philanderer, but they didn't know that the other women knew this. Here is a sketch of the general argument:

One philanderer: All women but one see one philanderer. That one sees none, so she knows and brands on night one. New information: The outsider directly gave new information.

Two philanderer: All women but two see two philanderers. These two each sees one. Each of these reasons that if her husband were not a philanderer, she would be in the previous case. But that is impossible after night one. Thus each brands on night two. New information: Each of the deceived wives knew there was one philanderer, but did not know that the other deceived wife knew there was at least one. This is new information.

Three philanderer: All women but three see three philanderers. These three deceived wives each sees two philanderers. Each of these reasons that if her husband were not a philanderer, we would be in the previous case. But that is impossible after night two. Thus each brands on night three. New information: Each of the deceived wives knew there were at least two philanderers, and hence knew that each of the others knew